Definable G homotopy extensions

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Abstract

Let G be a definably compact definable group, X a definable G set and Y a definable closed G subset of X. We prove that a pair (X, Y) admits a definable G homotopy extension.

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1 . Introduction.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$ and every definable map is assumed to be continuous unless otherwise stated.

General references on o-minimal structures are [1], [2], also see [6].

Let G be a definably compact definable group, X a definable G set and Y a definable G subset of X. We say that a pair (X, Y) *admits a definable G homotopy extension* if for any definable G map f from X to a definable G set Z and any definable G homotopy $F : Y \times [0, 1]_R \to Z$ with $F(y, 0) = f(y)$ for all $y \in Y$, there exists a definable G homotopy $H : X \times [0, 1]_R \to Z$ such that $H(x, 0) = f(x)$ for all $x \in X$ and $H|Y \times [0, 1]_R = F$, where $[0, 1]_R = \{x \in R | 0 \leq x \leq 1\}$.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the field $\mathbb{R}$ of real numbers. Definable G sets in $\mathcal{M}$ are studied in [5], [4], [3].

Theorem 1.1 ([4]). *If $\mathcal{N} = \mathcal{M}$, G is a compact definable group, X is a definable G set and Y is a definable closed G subset of X, then (X, Y) admits a definable G homotopy extension.*

In this paper, we generalize Theorem 1.1 to $\mathcal{N}$.

Theorem 1.2. *Let G be a definably compact definable group. If X is a definable G set and Y is a definable closed G subset of X, then (X, Y) admits a definable G homotopy extension.*

2 . Proof of Theorem 1.2

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is a *definable map* if the graph of f is definable.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

A definable set X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$

and for every definable map $f : (a, b)_R \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X, where $(a, b)_R = \{x \in R | a < x < b\}$.

If $R = \mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact definable set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is an *orthogonal G representation* if it is definable, where $O_n(R)$ means the nth orthogonal group of R. An *orthogonal G representation space* is R^n with the orthogonal action induced from an orthogonal G representation. A *definable G set* means a G invariant definable subset of some orthogonal G representation space.

A G map between definable G sets is a *definable G map* if it is a definable map.

Let X be a definable G set and Y a definable G subset of X. A *definable G retraction from X to Y* is a definable G map $r : X \to Y$ with $r|Y = id_Y$. A *definable strong G deformation retraction from X to Y* means a definable G map $F : X \times [0, 1]_R \to X$ such that $F(x, 0) = x$ for all $x \in X$, $F(y, t) = y$ for all $y \in Y, t \in [0, 1]_R$ and $F(X, 1) = Y$, where the action on $[0, 1]_R$ is trivial. Note that $F(\cdot, 1) : X \to Y$ is a definable G retraction from X to Y.

By a way similar to the proof of 3.4 [4], we have the following theorem.

Theorem 2.1. *Let G be a definably compact definable group and Y a definable closed G subset of a definable G set X. Then there exists a G invariant definable open neighborhood U of Y in X such that Y is a definable strong G deformation retract of both U and of the closure $\mathrm{cl}\, U$ of U in X.*

Proposition 2.2. *Let G be a definably compact definable group and A, B disjoint definable closed G subsets of a definable G set X. Then there exists a G invariant definable map $f : X \to [0, 1]_R$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Proof. By 10.2.8 [1], X/G is a definable set and the orbit map $\pi : X \to X/G$ is a definable map. Since A, B are definable and closed, $\pi(A), \pi(B)$ are definably closed sets. By 6.3.8 [1], there exists a definable map $h : X/G \to R$ with $\pi(A) = h^{-1}(0)$ and $\pi(B) = h^{-1}(1)$. Thus $f := h \circ \pi : X \to R$ is the required G invariant definable map. □

Proof of Theorem 1.2. To prove Theorem 1.2, we prove that there exists a definable G retraction $F : X \times [0, 1]_R \to (Y \times [0, 1]_R) \cup (X \times \{0\})$.

By Theorem 2.1, there exist a G invariant definable open neighborhood U of Y in X and a definable strong G deformation retraction $H : \mathrm{cl}\, U \times [0, 1]_R \to \mathrm{cl}\, U$ from $\mathrm{cl}\, U$ to Y. By Proposition 2.2, we have a G invariant definable map $\lambda : X \to [0, 1]_R$ with $\lambda^{-1}(0) = X - U$ and $\lambda^{-1}(1) = Y$. Put $B = \{(x, t) \in \mathrm{cl}\, U \times [0, 1]_R | \frac{1}{2} \le \lambda(x) < 1, 2(1 - \lambda(x)) \le t \le 1\}$, $C = \{(x, t) \in \mathrm{cl}\, U \times [0, 1]_R | \frac{1}{2} \le \lambda(x) < 1, 0 \le t \le 2(1 - \lambda(x))\}$, $D = \{(x, t) \in \mathrm{cl}\, U \times [0, 1]_R | 0 \le \lambda(x) \le \frac{1}{2}\}$, and $E = (X - U) \times [0, 1]_R$. Then B, C, D, E are G invariant definable subsets of $X \times [0, 1]_R$ such that $X \times [0, 1]_R = (Y \times [0, 1]_R) \cup B \cup C \cup D \cup E$, D and E are closed in $X \times [0, 1]_R$. The map $\psi : C \to [0, 1]_R$ defined by $\psi(x, t) = \frac{t}{2(1-\lambda(x))}$ is a G invariant definable function. Put $B' = B \cup (Y \times [0, 1]_R)$. We define a definable G retraction $F : X \times [0, 1]_R \to (Y \times [0, 1]_R) \cup (X \times \{0\})$,

$$F(x, t) = \begin{cases} (r(x), t - 2(1 - \lambda(x)), & (x, t) \in B' \\ (H(x, \psi(x, t)), 0), & (x, t) \in C \\ (H(x, 2t\lambda(x)), 0), & (x, t) \in D \\ (x, 0), & (x, t) \in E \end{cases},$$

where $r := H(\cdot, 1)$. Then F is a well-defined G map with the definable graph.

To prove continuity of F, it suffices to check that for a point y of Y, $F(x, t)$ converges to $(y, 0)$ if $(x, t) \in C$ and (x, t) tends to $(y, 0)$. Since H is continuous at (y, t), for any $\epsilon > 0$, there exists $\delta' > 0$ such that $||x - y|| < \delta', |t' - t| < \delta' \Rightarrow ||H(x, t') - H(y, t)|| < \epsilon$, where $||z||$ denotes the standard norm of z in a G representation containing X.

Fix $y \in Y, \epsilon > 0$. We define a function $\phi : [0,1]_R \to R, \phi(t) = \min\{\sup\{\delta' > 0 | ||x-y|| < \delta', |t'-t| < \delta' \Rightarrow ||H(x,t') - H(y,t)|| < \epsilon\}, 1\}$. Then ϕ is a positive function with the definable graph. By 3.1.2, 3.1.6 [1], there exist points $0 = a_0 < a_1 < \cdots < a_k = 1$ in $[0,1]_R$ such that for each j with $0 \le j \le k-1$, $\phi|(a_j, a_{j+1})_R$ is constant, or strictly monotone and continuous. Moreover $\lim_{x \to a_j+0} \phi(x)$ and $\lim_{x \to a_j-0} \phi(x)$ exist in R. By construction of ϕ, $\lim_{x \to a_j+0} \phi(x)$, $\lim_{x \to a_j-0} \phi(x)$ are positive. Thus modifying ϕ, if necessary, we may assume that for each j with $0 \le j \le k-1$, $\phi|[a_j, a_{j+1}]_R$ is a positive definable function. Since $[a_j, a_{j+1}]_R$ is definably compact, $\phi|[a_j, a_{j+1}]_R$ has the minimum $\delta_j > 0$. Let $\delta = \min\{\min_j \delta_j, \min_j \phi(a_j)\} > 0$. Then $||x-y|| < \delta \Rightarrow ||H(x,t) - y|| = ||H(x,t) - H(y,t)|| < \epsilon$ for any $t \in [0,1]_R$. Thus $F(x,t) \to (y,0)$ as $(x,t) \to (y,0)$. Note that $\lim_{(x,t)\to(y,0),(x,t)\in C} \psi(x,t)$ does not necessarily exist. $\square$

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